J. Differential Equations 250 (2011) 2715–2736

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Global well-posedness for the mass-critical nonlinear Schrödinger equation on $\mathbb{T}$

Yongsheng Li [a], Yifei Wu [a,*], Guixiang Xu [b]

[a] Department of Mathematics, South China University of Technology, Guangzhou, Guangdong, 510640, PR China
[b] Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing, 100088, PR China

ARTICLE INFO

Article history:
Received 26 November 2009
Revised 20 January 2011
Available online 2 February 2011

MSC:
primary 35Q53
secondary 47J35

Keywords:
Bourgain space
Schrödinger equation
Global well-posedness
I-method
Resonant decomposition

ABSTRACT

We consider the global well-posedness for the Cauchy problem of the mass-critical nonlinear Schrödinger equations in the periodic case. We show that it is globally well-posed in $H^s(\mathbb{T})$ for any $s > 2/5$. This improves the related work of Bourgain (2004) [2]. The key point is that we combine I-method with the resonant decomposition, which is developed in Colliander et al. (2008) [9], Li et al. (2011) [15], Miao et al. (2010) [16]. Another new ingredient here is that we obtain a bilinear Strichartz estimates in the periodic case which improves slightly the result given in De Silva et al. (2007) [11].

1. Introduction

In this paper, we consider the Cauchy problem for the periodic nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u - |u|^4 u = 0, & u : \mathbb{T} \times [0, T] \mapsto \mathbb{C}, \\ u(x, 0) = \phi(x) \in H^s(\mathbb{T}), \end{cases} \tag{1.1}$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the torus. The equation is mass-critical since the scaling

* Corresponding author.
E-mail addresses: yshli@scut.edu.cn (Y. Li), yerfmath@yahoo.cn (Y. Wu), xu_guixiang@iapcm.ac.cn (G. Xu).

doi:10.1016/j.jde.2011.01.025

$$u(x,t) \to \lambda^{-1/2} u\big(x/\lambda, t/\lambda^2\big), \quad \lambda > 0$$

leaves both the equation and the mass $\int_{\mathbb{T}} |u(x,t)|^2\,dx$ invariant.

The Cauchy problem (1.1) has been studied by several authors. It is known that (1.1) is local well-posedness in $H^s(\mathbb{T})$ for all $s > 0$, see Bourgain [1]. An immediate conclusion from [1] is the global well-posedness in $H^s(\mathbb{T})$ for $s \geqslant 1$, due to the Hamiltonian:

$$E(u) = \int_{\mathbb{T}} \frac{1}{2} u_x^2 + \frac{1}{6}|u|^6\,dx = E(\phi).$$

In [2], Bourgain combined the "norm form reduction" method with "I-method" to obtain the global well-posedness in $H^s(\mathbb{T})$ for $s > \frac{1}{2}$, moreover, by a refined trilinear Strichartz inequality, the author further extended the index to $s > s_*$ for some $s_* < \frac{1}{2}$. Based on the symplectic transformations, the "norm form reduction" method is to remove the strongly non-resonant part of Hamiltonian and thus reduce the nonlinearity to its "essential" part (in certain sense). The I-method, developed by Colliander, Keel, Staffilani, Takaoka and Tao (see [6,7] for examples), is based on the correction analysis of certain modified Hamiltonians and iteration. For the real line case, please refer to [3–5,7, 9,10,12,14,15,17,18], etc.

In this paper, the main result is

Theorem 1.1. *The Cauchy problem* (1.1) *is globally well-posed in $H^s(\mathbb{T})$ for $s > \frac{2}{5}$.*

The main idea is combing I-method with the resonant decomposition. Since the resonant interactions appears in the second correction term and the resonant set is large in this situation, it is hard to add the "correction term" to the modified energy $E(Iu)$ in a naive way (see [6,7] for examples). It is unlike the 1D cubic Schrödinger equation and the KdV equation cases. In these two cases the equations are completely integrable and the resonant sets are quite small and manageable. To overcome the resonant interactions, we make use of the resonant decomposition developed in [9,15,16]. More precisely, we will split the multiplier M_6 in the derivation of the first modified energy as

$$M_6 := \overline{M}_6 + \widetilde{M}_6,$$

where $\overline{M}_6$ contains some well-behaving term, and $\widetilde{M}_6$ contains the rest term and meanwhile it is "non-resonant". Therefore, on one hand we expect to give a better decay from the corresponding term on $\overline{M}_6$. On the other hand, since $\widetilde{M}_6$ is "non-resonant", we treat it by introducing a "correction-term" to the first generation modified energy and can also obtain a better decay from it. This method is analogous to the "norm form reduction" in [2] and the resonant decomposition in [9].

In addition, we shall use a slightly refinement of the bilinear Strichartz estimate obtained in [11]. Let us have a look at the real line case:

$$\left\| \int_{\xi=\xi_1+\xi_2} e^{ix\xi_1 - it\xi_1^2} e^{ix\xi_2 - it\xi_2^2} |\xi_1 - \xi_2|^{\frac{1}{2}} \widehat{\phi_1}(\xi_1)\widehat{\phi_2}(\xi_2)\,d\xi_1 \right\|_{L^2_{xt}} \lesssim \|\phi_1\|_{L^2} \|\phi_2\|_{L^2}. \tag{1.2}$$

(See [15] for example.) This bilinear Strichartz estimate presents a very useful smoothing effect in some special cases, especially when ϕ_1 and ϕ_2 have different frequent support. In this paper, we give a bilinear Strichartz estimates in the periodic case (see Proposition 2.1), which is corresponding to (1.2) in the periodic case and is an improvement version in [11].

The paper is organized as follows. In Section 2, we introduce some notations and state some preliminary estimates that will be used throughout this paper, especially, we prove the bilinear Strichartz estimates in this section. In Section 3, we prove a variant local well-posedness theory, review the I-method and obtain an upper bound on the increment of the modified energy. In Section 4, we give the proof of Theorem 1.1.

2. Notations and preliminary estimates

We use $A \lesssim B$ or $B \gtrsim A$ to denote the statement that $A \leqslant CB$ for some large constant C which may vary from line to line, and may depend on the data and the index s. When it is necessary, we will write the constants by $C_1, C_2, \ldots$ to see the dependency relationship. We use $A \sim B$ to mean $A \lesssim B \lesssim A$. We use $A \ll B$ to denote the statement $A \leqslant C^{-1}B$. Moreover, $a = O(b)$ means $|a| \lesssim |b|$; $a = o(b)$ means $|a| \ll |b|$. The notation $a+$ denotes $a + \epsilon$ for any small ϵ, and $a-$ for $a - \epsilon$. $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$, $D_x^\alpha = (-\partial_x^2)^{\alpha/2}$ and $J_x^\alpha = (1 - \partial_x^2)^{\alpha/2}$. We use $\|f\|_{L_x^p L_t^q}$ to denote the mixed norm $(\int \|f(x,\cdot)\|_{L^q}^p \, dx)^{\frac{1}{p}}$. Moreover, we denote $\mathscr{F}_x$ to be the Fourier transform corresponding to the variable x.

Now we introduce some other notations and definitions. We define $(dk)_\lambda$ to be the normalized counting measure on $\mathbb{T}_\lambda = \mathbb{R}/\lambda\mathbb{Z}$ such that

$$\int a(k)\,(dk)_\lambda = \frac{1}{\lambda} \sum_{k \in \frac{1}{\lambda}\mathbb{Z}} a(k).$$

Define the Fourier transform of a function f on $\mathbb{T}_\lambda$ by

$$\hat{f}(k) = \int_0^\lambda e^{-2\pi i x k} f(x)\,dx,$$

and thus the Fourier inversion formula

$$f(x) = \int e^{2\pi i x k} \hat{f}(k)\,(dk)_\lambda.$$

The usual properties of the Fourier transform hold

$$\|f\|_{L^2([0,\lambda])} = \|\hat{f}\|_{L^2((dk)_\lambda)} \quad \text{(Plancherel identity)}; \tag{2.3}$$

$$\int_0^\lambda f(x)\overline{g(x)}\,dx = \int \hat{f}(x)\overline{\hat{g}(x)}\,(dk)_\lambda \quad \text{(Parseval identity)}; \tag{2.4}$$

$$\widehat{fg}(k) = \hat{f} *_\lambda \hat{g}(k) = \int \hat{f}(k - k_1)\hat{g}(k_1)\,(dk)_\lambda \quad \text{(Convolution property)}. \tag{2.5}$$

We define the Sobolev space $H^s([0,\lambda])$ with the norm:

$$\|f\|_{H^s([0,\lambda])} = \big\| \langle k\rangle^s \hat{f}(k) \big\|_{L^2((dk)_\lambda)}.$$

For $s, b \in \mathbb{R}$, we define the Bourgain space for the λ-periodic Schrödinger equation to be the closure of the Schwartz class under the norm

$$\|u\|_{X^{\pm}_{s,b}} \equiv \left(\iint \langle k\rangle^{2s} \big\langle \tau \pm k^2 \big\rangle^{2b} \big|\hat{u}(k,\tau)\big|^2 \,(dk)_\lambda\, d\tau \right)^{1/2}. \tag{2.6}$$

In particular, we write $X_{s,b} \equiv X^+_{s,b}$. For an interval Ω, we define $X^{\Omega}_{s,b}$ to be the restriction of $X_{s,b}$ on

$\mathbb{R} \times \Omega$ with the norm

$$\|u\|_{X_{s,b}^{\Omega}} = \inf\bigl\{\|U\|_{X_{s,b}}\colon U|_{t\in\Omega} = u|_{t\in\Omega}\bigr\}. \tag{2.7}$$

When $\Omega = [-\delta, \delta]$, we write $X_{s,b}^{\Omega}$ as $X_{s,b}^{\delta}$.

Let $s < 1$ and $N \gg 1$ be fixed, we define the Fourier multiplier operator $I_{N,s}$ by

$$\widehat{I_{N,s}u}(k) = m_{N,s}(k)\hat{u}(k), \tag{2.8}$$

where the multiplier $m_{N,s}(k)$ is a smooth, monotone function satisfying $0 < m_{N,s}(k) \leqslant 1$ and

$$m_{N,s}(k) = \begin{cases} 1, & |k| \leqslant N, \\ N^{1-s}|k|^{s-1}, & |k| > 2N. \end{cases} \tag{2.9}$$

Sometimes we denote $I_{N,s}$ and $m_{N,s}$ as I and m respectively for short if there is no confusion.

It is obvious that the operator $I_{N,s}$ maps $H^s(\mathbb{T})$ into $H^1(\mathbb{T})$ for any $s < 1$. More precisely, there exists some positive constant C such that

$$C^{-1}\|u\|_{H^s} \leqslant \|I_{N,s}u\|_{H^1} \leqslant CN^{1-s}\|u\|_{H^s}. \tag{2.10}$$

Now we state some well-known Strichartz estimates. By the Strichartz estimate in [1] and rescaling, we have (see [11] for example for the proofs)

$$\|u\|_{L^4_{xt}([0,\lambda]\times\mathbb{R})} \lesssim \|u\|_{X^{\pm}_{0,\frac{3}{8}+}}, \tag{2.11}$$

and

$$\|u\|_{L^6_{xt}([0,\lambda]\times\mathbb{R})} \lesssim \lambda^{0+}\|u\|_{X^{\pm}_{0+,\frac{1}{2}+}}. \tag{2.12}$$

Moreover, Sobolev's embedding inequality implies that

$$\|u\|_{L^\infty_x L^\infty_t([0,\lambda]\times\mathbb{R})} \lesssim \|u\|_{X^{\pm}_{\frac{1}{2}+,\frac{1}{2}+}}. \tag{2.13}$$

Interpolating (2.12) and (2.13), we have

$$\|u\|_{L^q_{xt}([0,\lambda]\times\mathbb{R})} \lesssim \lambda^{0+}\|u\|_{X^{\pm}_{\theta(q)+,\frac{1}{2}+}}, \tag{2.14}$$

where $\theta(q) = \frac{1}{2} - \frac{3}{q}$ and $6 \leqslant q \leqslant \infty$.

Now we give some bilinear estimates. We write $S_\lambda(t)$ to be the solution map of the linear Schrödinger equation

$$i\partial_t u + \partial_{xx}u = 0, \quad x \in [0,\lambda],\ t \in \mathbb{R},$$

that is,

$$\widehat{S_\lambda(t)\phi}(k) = e^{-i(2\pi k)^2 t}\hat{\phi}(k).$$

Let $\eta(t)$ be a smooth bump function supported in the interval $[-2,2]$ such that $\eta(t)=1$ on $[-1,1]$. Define the Fourier integral operators $I_N^-(f,g)$ by

$$\widehat{I_N^-(f,g)}(k)=\int\limits_{k=k_1+k_2}\chi_{\{|k_1-k_2|\gtrsim N\}}\hat{f}(k_1)\hat{g}(k_2)\,(dk_1)_\lambda, \tag{2.15}$$

then we have

Proposition 2.1. *Let I_N^- be defined as* (2.15)*, then for any $\phi_1,\phi_2\in L^2([0,\lambda])$, we have*

$$\big\|I_N^-\big(\eta(t)S_\lambda(t)\phi_1,\eta(t)S_\lambda(t)\phi_2\big)\big\|_{L^2_{xt}}\lesssim C(N,\lambda)\|\phi_1\|_{L^2}\|\phi_2\|_{L^2}, \tag{2.16}$$

where

$$C(N,\lambda)=\begin{cases}1, & N\leqslant 1,\\ (\frac{1}{N}+\frac{1}{\lambda})^{\frac{1}{2}}, & N>1.\end{cases} \tag{2.17}$$

Proof. It is a slight modification of the proof in [11] (see also [7]), we present it briefly here. It follows easily by Hölder and (2.11) when $N\lesssim 1$. So we only consider it when $N\gg 1$. Then, by Plancherel's identity, the left-hand side of (2.16) is reduced to

$$\left\|\int\limits_{\substack{k_1+k_2=k\\ \tau_1+\tau_2=\tau}}\chi_{\{|k_1-k_2|\gtrsim N\}}\hat{\eta}\big(\tau_1-k_1^2\big)\hat{\eta}\big(\tau_2-k_2^2\big)\phi_1(k_1)\phi_2(k_2)\,(dk_1)_\lambda\,d\tau\right\|_{L^2((dk)_\lambda\,d\tau)}.$$

Let $\psi=\hat{\eta}*\hat{\eta}$, then by Hölder's inequality it further turns to

$$\left\|\int\limits_{k_1+k_2=k}\chi_{\{|k_1-k_2|\gtrsim N\}}\psi\big(\tau-k_1^2-k_2^2\big)\,(dk_1)_\lambda\right\|_{L^\infty((dk)_\lambda\,d\tau)}^{\frac{1}{2}}\|\phi_1\|_{L^2}\|\phi_2\|_{L^2}.$$

Therefore, we only need to show

$$M:=\left\|\int\limits_{k_1+k_2=k}\chi_{\{|k_1-k_2|\gtrsim N\}}\psi\big(\tau-k_1^2-k_2^2\big)\,(dk_1)_\lambda\right\|_{L^\infty((dk)_\lambda\,d\tau)}\lesssim C(N,\lambda)^2. \tag{2.18}$$

On the other hand,

$$M\lesssim\frac{1}{\lambda}\#A, \tag{2.19}$$

here and in the sequel # denotes the number of elements in the set, and

$$\begin{aligned}A&=\left\{k_1\in\frac{1}{\lambda}\mathbb{Z}\colon\ k_2=k-k_1,\ |k_1-k_2|\gtrsim N,\ \tau-k_1^2-k_2^2=O(1)\right\}\\ &=\left\{k_1\in\frac{1}{\lambda}\mathbb{Z}\colon\ k_2=k-k_1,\ |k_1-k_2|\gtrsim N,\ (k_1-k_2)^2=2\tau-k^2+O(1)\right\} && (2.20)\\ &=\left\{k_1\in\frac{1}{\lambda}\mathbb{Z}\colon\ k_2=k-k_1,\ |k_1-k_2|\gtrsim N,\ k_1=\frac{1}{2}k+\frac{1}{2}\sqrt{a+O(1)}\right\}, && (2.21)\end{aligned}$$

where we set $a = 2\tau - k^2$ above and thus one has $|a| \gtrsim N^2$ by (2.20). For any $x_1, x_2 \in A$, by (2.21) we have

$$|x_1 - x_2| = |\sqrt{a+\varepsilon_1} - \sqrt{a+\varepsilon_2}| = \frac{|\varepsilon_1 - \varepsilon_2|}{\sqrt{a+\varepsilon_1} + \sqrt{a+\varepsilon_2}} \lesssim \frac{1}{N},$$

where $\varepsilon_1, \varepsilon_2 = O(1)$. This implies that A belongs to a set of length $\frac{1}{N}$, and thus

$$\#A \lesssim \frac{\lambda}{N} + 1.$$

Then by (2.18) and (2.19), we have the claim. □

Remark 2.1. Note that (2.16) becomes exactly (1.2) when $\lambda \to \infty$ (at least when $N \geqslant 1$).

As a corollary, we have another bilinear Strichartz estimate. Let the integral operators $I_N^+(f, g)$ denote

$$\widehat{I_N^+(f,g)}(k) = \int\limits_{k=k_1+k_2} \chi_{\{|k_1+k_2| \gtrsim N\}} \hat{f}(k_1)\hat{g}(-k_2)\,(dk_1)_\lambda, \tag{2.22}$$

then we have

Corollary 2.1. *Let I_N^+ be defined as* (2.22)*, then for any $\phi_1, \phi_2 \in L^2([0,\lambda])$,*

$$\big\| I_N^+\big(\eta(t)S_\lambda(t)\phi_1, \eta(t)S_\lambda(t)\phi_2\big)\big\|_{L^2_{xt}} \lesssim C(N,\lambda)\|\phi_1\|_{L^2}\|\phi_2\|_{L^2}, \tag{2.23}$$

where $C(N,\lambda)$ is the same as (2.17)*.*

Proof. By the process above, we only need to show

$$\frac{1}{\lambda}\#\left\{k_1 \in \frac{1}{\lambda}\mathbb{Z}\colon\ k_2 = k - k_1,\ |k_1 - k_2| \gtrsim N,\ k_1 = \frac{1}{2}k + \frac{1}{2}\sqrt{a + O(1)}\right\} \lesssim C(N,\lambda)^2. \tag{2.24}$$

On the other hand,

$$\begin{aligned}
&\#\left\{k_1 \in \frac{1}{\lambda}\mathbb{Z}\colon\ k_2 = k - k_1,\ |k_1 + k_2| \gtrsim N,\ \tau - k_1^2 + k_2^2 = O(1)\right\} \\
&\quad = \#\left\{k_1 \in \frac{1}{\lambda}\mathbb{Z}\colon\ k_2 = k - k_1,\ |k| \gtrsim N,\ k_1 = \frac{\tau + k^2 + O(1)}{2k}\right\} \lesssim \frac{\lambda}{N} + 1.
\end{aligned}$$

Then the claim follows by combining the above and (2.24). □

Proposition 2.2. *Let u, v be the λ-periodic functions of x, and $\mathrm{supp}_t\, u(t,x)$, $\mathrm{supp}_t\, v(t,x) \subset [-\lambda,\lambda]$, then for the operators $I_N^\pm$ defined in* (2.15), (2.22)*, we have*

$$\big\| I_N^\pm(u,v)\big\|_{L^2_{xt}} \lesssim C(N,\lambda)\|u\|_{X^\pm_{0,\frac{1}{2}+}}\|v\|_{X^\pm_{0,\frac{1}{2}+}}. \tag{2.25}$$

3. Global well-posedness

3.1. Rescaling

Our aim of this paper is to construct the 1-periodic solution of (1.1) on arbitrary time interval $[0, T]$. Write

$$u_\lambda(x,t) = \lambda^{-\frac{1}{2}} u\big(x/\lambda, t/\lambda^2\big); \qquad \phi_\lambda(x) = \lambda^{-\frac{1}{2}}\phi(x/\lambda),$$

then u_λ satisfies

$$\begin{cases} i\partial_t u_\lambda + \partial_x^2 u_\lambda - |u_\lambda|^4 u_\lambda = 0, & u_\lambda : [0,\lambda]\times[0,T] \mapsto \mathbb{C}, \\ u_\lambda(x,0) = \phi_\lambda(x). \end{cases} \tag{3.26}$$

Moreover, the solution u of (1.1) exists on $[0, T]$ if and only if u_λ exists on $[0, \lambda^2 T]$. By (2.10) and $m(k) \leqslant 1$, we get that

$$\|I\phi_\lambda\|_{L_x^2} \leqslant \|\phi_\lambda\|_{L_x^2} = \|\phi\|_{L_x^2}; \qquad \|\partial_x I\phi_\lambda\|_{L^2} \lesssim N^{1-s}/\lambda^s \cdot \|\phi\|_{H^s}.$$

Hence, if we choose

$$\lambda \sim N^{\frac{1-s}{s}}, \tag{3.27}$$

then we have

$$\|I\phi_\lambda\|_{H^1} \lesssim 1. \tag{3.28}$$

From now on, we consider u_λ instead of u, and our task is to construct the λ-periodic solution of (3.26) on $[0, \lambda^2 T]$. For simplicity, we drop the subscript λ until the last section.

3.2. A variant local well-posedness

In this subsection, we will give a variant local well-posedness result as follows. We do not try to obtain the sharp result here (one may also refer to [11]). Recall that we write u_λ as u, and ϕ_λ as ϕ for simplicity.

Lemma 3.1. *For any $s > \frac{1}{8}$, $0 < \delta < 1$ and λ-periodic function u, we have*

$$\big\|\eta(t/\delta)|u|^4 u\big\|_{X_{s,-\frac{1}{2}+}} \lesssim \lambda^{0+}\delta^{\frac{1}{8}-}\|u\|_{X_{s,\frac{1}{2}+}}^5. \tag{3.29}$$

Proof. First for any $s \in \mathbb{R}$, $-\frac{1}{2} < b' \leqslant b \leqslant 0$, we have (see [13] for example)

$$\big\|\eta(t/\delta) f\big\|_{X_{s,b'}} \lesssim \delta^{b-b'}\big\|\eta(t/\delta) f\big\|_{X_{s,b}},$$

which, together with the duality of (2.11), implies that

$$\big\|\eta(t/\delta) f\big\|_{X_{s,-\frac{1}{2}+}} \lesssim \delta^{\frac{1}{8}-}\big\|\eta(t/\delta) f\big\|_{X_{s,-\frac{3}{8}-}} \lesssim \delta^{\frac{1}{8}-}\|f\|_{L_{xt}^{\frac{4}{3}}}. \tag{3.30}$$

By Hölder's inequality and (2.14), we obtain

$$\left\| J_x^s\left(|u|^4 u\right)\right\|_{L_{xt}^{\frac{4}{3}}} \lesssim \left\| J_x^s u\right\|_{L_{xt}^4} \|u\|_{L_{xt}^8}^4 \lesssim \lambda^{0+}\|u\|_{X_{s,\frac{1}{2}+}}^5. \tag{3.31}$$

This completes the proof of the lemma. $\square$

By the standard iteration argument and Lemma 12.1 in [8], we have the following local result.

Proposition 3.1. *Let $s > \frac{1}{8}$, then the Cauchy problem (3.26) is locally well-posed for the initial data ϕ satisfying $I\phi \in H^1(\mathbb{T})$. Moreover, the solution exists on the interval $[0, \delta]$ with the lifetime*

$$\delta \sim \lambda^{-\epsilon}\|I_{N,s}u_0\|_{H^1}^{-\mu} \tag{3.32}$$

for any small $\epsilon > 0$ and some $\mu > 0$. Further, the solution satisfies the estimate

$$\|Iu\|_{X^\delta_{1,\frac{1}{2}+}} \lesssim \|I\phi\|_{H^1}. \tag{3.33}$$

3.3. Modified energies and the I-method

From now on, let u be the solution of (3.26). For an even integer n and an n-multiplier $M_n(k_1,\dots,k_n)$ defined on the hyperplane

$$\Gamma_n = \left\{(k_1,\dots,k_n)\colon k_1+\cdots+k_n = 0\right\}, \tag{3.34}$$

we define the quantity

$$\Lambda_n(M_n; f_1,\dots,f_n) \equiv \int\limits_{\Gamma_n} M_n(k_1,\dots,k_n)\prod_{j=1}^n \mathscr{F}_x f_j(k_j,t)\,(dk_1)_\lambda\cdots(dk_{n-1})_\lambda,$$

and adopt the notation $\Lambda_n(M_n) = \Lambda_n(M_n; u,\bar{u},\dots,u,\bar{u})$. Then by (3.26) and a simple computation, we have

$$\frac{d}{dt}\Lambda_n(M_n) = \Lambda_n(M_n\alpha_n) + i\Lambda_{n+4}\left(\sum_{j=1}^n (-1)^j X_j(M_n)\right), \tag{3.35}$$

where

$$\alpha_n = i\sum_{j=1}^n (-1)^j k_j^2; \qquad X_j(M_n) = M_n(k_1,\dots,k_{j-1},k_j+\cdots+k_{j+4},k_{j+5},\dots,k_{n+4}).$$

Define the first generation modified energy as

$$\begin{aligned} E_I^1\big(u(t)\big) &:= E(Iu) = \frac{1}{2}\left\|\partial_x Iu(t)\right\|_{L^2}^2 + \frac{1}{6}\left\|Iu(t)\right\|_{L^6}^6 \\ &= \Lambda_2(\sigma_2) + \Lambda_6(\sigma_6), \end{aligned} \tag{3.36}$$

where

$$\sigma_2 := -\frac{1}{2} m(k_1)k_1 m(k_2)k_2; \qquad \sigma_6 := \frac{1}{6} m(k_1)\cdots m(k_6).$$

Then by (3.35) and note that $\alpha_2 = 0$ if $k_1 + k_2 = 0$, we have

$$\frac{d}{dt} E_I^1\big(u(t)\big) = \Lambda_6(M_6) + \Lambda_{10}(M_{10}), \tag{3.37}$$

where

$$M_6(k_1,\dots,k_6) := \frac{i}{6}\sum_{j=1}^{6}(-1)^{(j+1)} m^2(k_j)k_j^2 + \sigma_6\alpha_6 := M_6^1 + M_6^2;$$

$$M_{10}(k_1,\dots,k_{10}) := i\sum_{j=1}^{6}(-1)^j X_j(\sigma_6).$$

We adopt the notation that

$$\big|k_1^*\big| \geqslant \big|k_2^*\big| \geqslant \cdots \geqslant \big|k_6^*\big| \geqslant \cdots \geqslant \big|k_{10}^*\big|.$$

It is obvious that $M_6, M_{10} = 0$ if $|k_1^*| \ll N$. Thus we may assume

$$\big|k_1^*\big| \sim \big|k_2^*\big| \gtrsim N.$$

Moreover, for M_6, by the symmetry, we may restrict further

$$|k_1| \geqslant |k_3| \geqslant |k_5|, \qquad |k_2| \geqslant |k_4| \geqslant |k_6|, \quad \text{and} \quad |k_1| \geqslant |k_2|.$$

Hence under these assumptions, we have $k_1^* = k_1$, $k_2^* = k_2$ or k_3. Now denote

$$\begin{aligned}
\Upsilon &= \big\{(k_1,\dots,k_6)\in\Gamma_6\colon\ \big|k_1^*\big| \sim \big|k_2^*\big| \gtrsim N\big\};\\
\Omega_1 &= \big\{(k_1,\dots,k_6)\in\Upsilon\colon\ |k_1| \gg |k_2|\big\};\\
\Omega_2 &= \big\{(k_1,\dots,k_6)\in\Upsilon\colon\ \big|k_3^*\big| \gg \big|k_4^*\big|\big\};\\
\Omega_3 &= \big\{(k_1,\dots,k_6)\in\Upsilon\colon\ |k_1| \sim |k_5|,\ |k_5| \gg |k_4|\big\};\\
\Omega_4 &= \big\{(k_1,\dots,k_6)\in\Upsilon\colon\ |k_1| \sim |k_6|,\ |k_6| \gg |k_3|, \text{ and}\\
&\qquad \text{either } |k_1| = |k_2| + o\big(|k_1|\big) \text{ or } k_2\cdot k_4 > 0,\ k_2\cdot k_6 > 0\big\};\\
\Omega_5 &= \big\{(k_1,\dots,k_6)\in\Upsilon\colon\ |k_1| \sim |k_2| \gtrsim N \gg \big|k_3^*\big|,\ \big|k_1^2 - k_2^2\big| \gg \big|k_3^2 - k_4^2 + k_5^2 - k_6^2\big|\big\}.
\end{aligned}$$

Rewrite (3.37) by

$$\frac{d}{dt} E_I^1\big(u(t)\big) = \Lambda_6(\overline{M}_6) + \Lambda_6(\widetilde{M}_6) + \Lambda_{10}(M_{10}),$$

where

$$\overline{M}_6 = (\chi_{\Gamma_6} - \chi_{\Omega}) M_6^1 + (\chi_{\Gamma_6} - \chi_{\Upsilon}) M_6^2; \qquad \widetilde{M}_6 = \chi_{\Omega} M_6^1 + \chi_{\Upsilon} M_6^2, \tag{3.38}$$

and

$$\Omega = \bigcup_{j=1}^{5} \Omega_j, \quad \text{and} \quad \chi \text{ is the characterization function of the set.}$$

Note that Ω is the non-resonant region, while Υ_6/Ω is the resonant region.

Define the second generation modified energy $E_I^2(u(t))$ by

$$E_I^2\big(u(t)\big) = E_I^1\big(u(t)\big) + \Lambda_6(\widetilde{\sigma}_6), \qquad \widetilde{\sigma}_6 = -\widetilde{M}_6/\alpha_6. \tag{3.39}$$

Then we have

$$\frac{d}{dt} E_I^2\big(u(t)\big) = \Lambda_6(\overline{M}_6) + \Lambda_{10}(\overline{M}_{10}), \tag{3.40}$$

where

$$\overline{M}_{10} = i \sum_{j=1}^{6} (-1)^j \big(X_j^4(\sigma_6) + X_j^4(\widetilde{\sigma}_6) \big).$$

Remark 3.1. As shown in [15], we describe the resonant decomposition again. A nature choice of the second modified energy is $E_I^2(u(t)) = \Lambda_6(\check{\sigma}_6) + E_I^1(u(t))$ with $\check{\sigma}_6 = -M_6/\alpha_6$, but unfortunately, $\check{\sigma}_6$ is singular and hard to handle. The strategy used here is to split the M_6 into two part: $\overline{M}_6$ and $\widetilde{M}_6$. To see clearly the properties of two parts, we only consider the situation of $|k_2^*| \gtrsim N \gg |k_3^*|$. On one hand, we see that $\overline{M}_6 \lesssim |k_3^*||k_4^*|$ (see Lemma 3.4(ii) below; compared to M_6, we may only get: $|M_6| \lesssim |k_1^*||k_3^*|$), which is a relatively low frequency term. On the other hand, we have $|\widetilde{M}_6| \lesssim |\alpha_6|$, which is referred to be non-resonant. Therefore, by defining $E_I^2(u(t))$ as (3.39), we may treat M_6 reasonably with remaining $\overline{M}_6$ and deducing $\widetilde{M}_6$ to a part of $\overline{M}_{10}$, which is a higher order cancellation.

Lemma 3.2. *$|\widetilde{M}_6| \lesssim |\alpha_6|$, i.e. $|\widetilde{\sigma}_6| \lesssim 1$.*

Proof. It is much similar to Lemma 3.1 in [15]. Since $|M_2^6/\alpha_6| = |\sigma_6| \lesssim 1$, it suffices to show that

$$\big|\chi_{\Omega_j} M_6^1\big| \lesssim |\alpha_6|, \quad \text{for } j = 1, \dots, 5, \tag{3.41}$$

that is, for $(k_1, \dots, k_6) \in \Omega$,

$$\big|m^2(k_1)k_1^2 - m^2(k_2)k_2^2 + \dots + m^2(k_5)k_5^2 - m^2(k_6)k_6^2\big| \lesssim \big|k_1^2 - k_2^2 + \dots + k_5^2 - k_6^2\big|.$$

First, we have

$$\big|M_6^1\big| \lesssim k_1^2. \tag{3.42}$$

In Ω_1, $|k_1| \ll |k_2|$, so we have

$$|\alpha_6| \sim |k_1|^2. \tag{3.43}$$

Then (3.41) for $j = 1$ follows from (3.42).

In $\Omega_2\backslash\Omega_1$, we have $k_1^*+k_2^*+k_3^*=o(|k_1^*|)$, thus

$$k_1^*\cdot k_2^*<0,\qquad k_2^*\cdot k_3^*>0, \tag{3.44}$$

which implies that

$$\left|k_1^*\right|=\left|k_2^*\right|+\left|k_3^*\right|+o\left(\left|k_3^*\right|\right). \tag{3.45}$$

If $k_2^*=k_2$, $k_3^*=k_3$, then by (3.44) and (3.45),

$$\begin{aligned}|\alpha_6|&=k_1^2-k_2^2+k_3^2+o\left(k_3^2\right)\\&=2k_2\cdot k_3+2k_3^2+o\left(k_3^2\right)\sim k_2\cdot k_3.\end{aligned} \tag{3.46}$$

For $\widetilde{M}_6^1$, we first claim that, for any $k,k'\in\mathbb{R}$, $|k|\geqslant|k'|$,

$$\left|m^2(k)k^2-m\left(k'\right)k'^2\right|\lesssim m^2(k)\left(k^2-k'^2\right). \tag{3.47}$$

Indeed, if $|k|\gg|k'|$, then it follows by triangle's inequality. While if $|k|\sim|k'|$, it follows from the mean value theorem. By (3.47) and similar estimate as in (3.46), we have

$$\begin{aligned}\left|\chi_{\Omega_2\backslash\Omega_1}M_6^1\right|&\leqslant\left|m^2(k_1)k_1^2-m^2(k_2)k_2^2\right|+|k_3|^2+\cdots+|k_6|^2\\&\lesssim\left|k_1^2-k_2^2\right|+|k_3|^2\sim k_2\cdot k_3.\end{aligned} \tag{3.48}$$

Then (3.41) in this case follows by combining (3.46) with (3.48).

If $k_2^*=k_2$, $k_3^*=k_4$, then by (3.45), we have

$$k_1^2-k_2^2-k_4^2=\left(k_2+k_4+o(k_4)\right)^2-k_2^2-k_4^2\sim k_2\cdot k_4,$$

which implies that

$$|\alpha_6|=\left(k_1^2-k_2^2-k_4^2\right)+o\left(|k_4|^2\right)\sim k_2\cdot k_4.$$

Similar to (3.48), we have $|\chi_{\Omega_2\backslash\Omega_1}M_6^1|\sim k_2\cdot k_4$ in this case. So we have the desired result in (3.41) in this situation.

If $k_2^*=k_3$, then $k_3^*=k_2$ (since $|k_1|\sim|k_2|$ in $\Omega_2\backslash\Omega_1$). In this case, we have $k_3^2\geqslant k_2^2$, which yields that

$$|\alpha_6|=\left(k_1^2+k_3^2-k_2^2\right)+o\left(|k_2|^2\right)\geqslant k_1^2+o\left(|k_2|^2\right)\sim k_1^2.$$

Then the desired result in (3.41) in this case follows from (3.42). To sum up, we obtain (3.41) for $j=2$.

In $\Omega_3\backslash\Omega_1$, we have $\{k_1^*,k_2^*,k_3^*,k_4^*\}=\{k_1,k_2,k_3,k_5\}$ and $|k_1|\sim|k_2|\sim|k_3|\sim|k_5|$. Since $k_1^2\geqslant k_2^2$, we have

$$|\alpha_6|=\left(k_1^2-k_2^2+k_3^2+k_5^2\right)+o\left(|k_1|^2\right)\geqslant\left(k_3^2+k_5^2\right)\sim|k_1|^2.$$

Then (3.41) for $j=3$ follows from (3.42) again.

In $\Omega_4 \backslash \Omega_1$, we have $\{k_1^*, k_2^*, k_3^*, k_4^*\} = \{k_1, k_2, k_4, k_6\}$ and $|k_1| \sim |k_2| \sim |k_4| \sim |k_6|$. By the definition, we split this domain into the following two cases:

$$|k_1| = |k_2| + o\big(|k_1|\big); \qquad k_2 \cdot k_4 > 0, \qquad k_2 \cdot k_6 > 0.$$

If $|k_1| = |k_2| + o(|k_1|)$, then $k_1^2 - k_2^2 = o(|k_1|^2)$. Therefore, we have

$$|\alpha_6| \geqslant \big(k_4^2 + k_6^2\big) - \big|k_1^2 - k_2^2\big| - \big|k_3^2 + k_5^2\big| = \big(k_4^2 + k_6^2\big) + o\big(|k_1|^2\big) \sim |k_1|^2. \tag{3.49}$$

If $k_2 \cdot k_4 > 0$, $k_2 \cdot k_6 > 0$, then we have $|k_1| = |k_2| + |k_4| + |k_6| + o(|k_6|)$, which implies that

$$\begin{aligned} k_1^2 - k_2^2 - k_4^2 - k_6^2 &= \big(k_2 + k_4 + k_6 + o(k_4)\big)^2 - k_2^2 - k_4^2 - k_6^2 \\ &= 2(k_2 \cdot k_4 + k_2 \cdot k_6 + k_4 \cdot k_6) + o\big(|k_1|^2\big) \sim |k_1|^2. \end{aligned}$$

Therefore, we have

$$|\alpha_6| = \big(k_1^2 - k_2^2 - k_4^2 - k_6^2\big) + o\big(|k_1|^2\big) \sim |k_1|^2. \tag{3.50}$$

Then (3.41) for $j = 4$ follows from (3.42), (3.49) and (3.50).

In Ω_5, we have

$$|\alpha_6| \sim \big|k_1^2 - k_2^2\big|.$$

By (3.47), we obtain

$$\begin{aligned} \big|\chi_{\Omega_5} M_6^1\big| &\lesssim \big|m^2(k_1)k_1^2 - m^2(k_2)k_2^2\big| + \big|k_3^2 - k_4^2 + k_5^2 - k_6^2\big| \\ &\lesssim \big|k_1^2 - k_2^2\big| + \big|k_3^2 - k_4^2 + k_5^2 - k_6^2\big| \sim \big|k_1^2 - k_2^2\big|. \end{aligned}$$

So we have (3.41) for $j = 5$. This completes the proof of the lemma. $\square$

Now we give the comparison between $E_I^1(u(t))$ and $E_I^2(u(t))$.

Lemma 3.3. *For any $s > \frac{1}{3}$, we have*

$$\big|\Lambda_6(\widetilde{\sigma}_6)(t)\big| \lesssim N^{0-} \, \big\|Iu(t)\big\|_{H_x^1}^6.$$

Proof. By Lemma 3.1, it suffices to show

$$\int\limits_{\Gamma_6} \frac{\mathscr{F}_x f_1(k_1, t)\overline{\mathscr{F}_x f_2}(-k_2, t) \cdots \overline{\mathscr{F}_x f_6}(-k_6, t)}{\langle k_1\rangle m(k_1) \cdots \langle k_6\rangle m(k_6)} \lesssim N^{0-} \, \|f_1\|_{L_x^2} \cdots \|f_6\|_{L_x^2}. \tag{3.51}$$

Without loss of generality, we may assume that $\{k_1, k_2\} = \{k_1^*, k_2^*\}$. Note that $|k_1^*| \sim |k_2^*| \gtrsim N$ and

$$\langle k\rangle m(k) = \langle k\rangle, \quad \text{for } |k| \leqslant N; \qquad \langle k\rangle m(k) \sim N^{1-s}|k|^s, \quad \text{for } |k| \gtrsim N,$$

then by Hölder's inequality, the left-hand side of (3.51) is bounded by

$$N^{0-}\|f_1(t)\|_{L_x^2}\|f_2(t)\|_{L_x^2}\|J_x^{-\frac{1}{2}-}f_3(t)\|_{L_x^\infty}\cdots\|J_x^{-\frac{1}{2}-}f_6(t)\|_{L_x^\infty}.$$

Hence we have the result by Sobolev's inequality. □

3.4. An upper bound on the increment of $E_I^2(u(t))$

To bound the increment of $E_I^2(u(t))$ is the key ingredient in the lower regularity theory. By (3.40), it suffices to obtain the related 6-linear and 10-linear estimates.

First, we establish the following pointwise estimates on the multiplier $\overline{M}_6$.

Lemma 3.4. *The multiplier $\overline{M}_6$ is defined in* (3.38)*, then, we have*

(i) $|\overline{M}_6|\lesssim m(k_1^*)m(k_3^*)|k_1^*||k_3^*|$.
(ii) *If* $|k_1^*|\sim|k_2^*|\gtrsim N\gg|k_3^*|$, *then* $|\overline{M}_6|\lesssim|k_3^*||k_4^*|$.
(iii) *If* $|k_1+k_2|\lesssim|k_5+k_6|$, *then* $|\overline{M}_6|\lesssim m(k_1)|k_1|\cdot\max\{|k_5|,|k_6|\}$.
(iv) *If* $|k_1+k_2|\gg|k_5+k_6|$, *then* $|\overline{M}_6|\lesssim m(k_1)|k_1||k_1+k_2|$.

Proof. Since $\overline{M}_6=0$ when $|k_1|,\dots,|k_6|\ll N$, we may assume that $(k_1,\dots,k_6)\in\Upsilon$. Thus we have

$$\overline{M}_6=(\chi_{\Gamma_6}-\chi_{\Omega})M_6^1.$$

For (i), if $|k_1^*|\sim|k_3^*|$, since $m(k)|k|$ is increasing in k, and thus we have

$$m(k_j)|k_j|\leqslant m(k_1)|k_1|,\quad\text{for } j=1,\dots,6, \tag{3.52}$$

which implies that

$$\overline{M}_6\leqslant 6m^2(k_1)k_1^2\sim m(k_1^*)m(k_3^*)|k_1^*||k_3^*|.$$

Now we consider the case: $|k_1^*|\gg|k_3^*|$. Since $|k_1|\sim|k_2|$ in $\Gamma_6\backslash\Omega_1$, we have $\{k_1^*,k_2^*\}=\{k_1,k_2\}$. Therefore, by (3.47) and (3.52), we obtain

$$\begin{aligned}|\overline{M}_6|&\lesssim|m^2(k_1)k_1^2-m^2(k_2)k_2^2|+m^2(k_3^*)|k_3^*|^2\\&\lesssim m^2(k_1)|k_1+k_2||k_1-k_2|+m^2(k_3^*)|k_3^*|^2\\&\lesssim m(k_1)m(k_3^*)|k_1||k_3^*|.\end{aligned}$$

This implies (i).

For (ii), under the assumption of (ii), it holds that $\{k_1^*,k_2^*\}=\{k_1,k_2\}$. Then for $(k_1,\dots,k_6)\in\Gamma_6\backslash\Omega_5$, by (3.47) we have

$$\begin{aligned}|\overline{M}_6|&\lesssim|m^2(k_1)k_1^2-m^2(k_2)k_2^2|+|k_3^2-k_4^2+k_5^2-k_6^2|\\&\lesssim|k_1^2-k_2^2|+|k_3^2-k_4^2+k_5^2-k_6^2|\\&\lesssim|k_3^2-k_4^2+k_5^2-k_6^2|\lesssim|k_3^*|^2.\end{aligned}$$

Hence we get (ii) by the fact that $|k_3^*|\sim|k_4^*|$ in $\Gamma_6\backslash\Omega_2$.

For (iii), we also have $|k_3+k_4| \lesssim |k_5+k_6| \lesssim |k_5^*|$, thus by (3.47) and (3.52), we have

$$\begin{aligned}|\overline{M}_6| &\lesssim \left|m^2(k_1)k_1^2 - m^2(k_2)k_2^2\right| + \left|m^2(k_3)k_3^2 - m^2(k_4)k_4^2\right| + \left|m^2(k_5)k_5^2 - m^2(k_6)k_6^2\right| \\ &\lesssim m^2(k_1)|k_1||k_1+k_2| + m(k_1)|k_1||k_3+k_4| + m(k_1)|k_1||k_5+k_6| \\ &\lesssim m(k_1)|k_1||k_5+k_6| \\ &\lesssim m(k_1)|k_1|\max\{|k_5|,|k_6|\}. \end{aligned} \tag{3.53}$$

For (iv), note that $|k_1+k_2| \sim |k_3+k_4| \gg |k_5+k_6|$, thus by the similar estimate as in (3.53), we have

$$\begin{aligned}|\overline{M}_6| &\lesssim m^2(k_1)|k_1||k_1+k_2| + m(k_1)|k_1||k_3+k_4| + m(k_1)|k_1||k_5+k_6| \\ &\sim m(k_1)|k_1||k_1+k_2|. \end{aligned}$$

This completes the proof of the lemma. □

Next, we aim to give the 6-linear and 10-linear estimates. Before doing this, we adopt a notation. We fix the number λ as (3.27) and define

$$\alpha(s) = \frac{1}{2}\min\{(1-s)/s, 1\}. \tag{3.54}$$

Then the constant $C(N,\lambda)$ defined in (2.17) satisfies

$$C(N,\lambda) = N^{-\alpha(s)}, \quad \text{when } N \geqslant 1.$$

Proposition 3.2. *For any $s \geqslant \frac{1}{3}$, $\delta \in (0,1)$ and $\alpha(s)$ defined in* (3.54)*, we have*

$$\left|\int_0^\delta \Lambda_6(\overline{M}_6)\,dt\right| \lesssim N^{-2-2\alpha(s)+}\lambda^{0+}\,\|Iu\|^6_{X^\delta_{1,\frac{1}{2}+}}. \tag{3.55}$$

Proof. Since $\overline{M}_6 = 0$ for $|k_1|,\dots,|k_6| \leqslant N$, we may assume that $|k_1^*| \sim |k_2^*| \gtrsim N$. To extend the integration domain from $[0,\delta]$ to $\mathbb{R}$, we may need to borrow $|k_1^*|^{0-}$ from the multiplier (see [6] for the argument), but this will not be mentioned since it will only be recorded by N^{0+} at the end. Therefore, by Plancherel's identity and the fact $\bar{\hat{f}}(k,\tau) = \hat{\bar{f}}(-k,-\tau)$, we only need to show

$$\begin{aligned}&\int_{\Gamma_6^2} \frac{\overline{M}_6(k_1,\dots,k_6)\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_6}(k_6,\tau_6)}{\langle k_1\rangle m(k_1)\langle k_6\rangle m(k_6)} \\ &\quad\lesssim N^{-2-2\alpha(s)+}\lambda^{0+}\,\|f_1\|_{X^+_{0,\frac{1}{2}+}}\|f_2\|_{X^-_{0,\frac{1}{2}+}}\cdots\|f_5\|_{X^+_{0,\frac{1}{2}+}}\|f_6\|_{X^-_{0,\frac{1}{2}+}}, \end{aligned} \tag{3.56}$$

where the set $\Gamma_6^2 = \{(k,\tau):\ k=(k_1,\dots,k_6),\ \tau=(\tau_1,\dots,\tau_6),\ k_1+\dots+k_6=0,\ \tau_1+\dots+\tau_6=0\}$. Without loss of generality, we may assume that $\widehat{f_j}$, $j=1,\dots,6$, are real positive functions. Now we divide Γ_6^2 into four regions:

$$A_1 = \{(k,\tau) \in (\Gamma_6 \backslash \Omega) \times \Gamma_6\colon\ |k_2^*| \gtrsim N \gg |k_3^*|\};$$
$$A_2 = \{(k,\tau) \in (\Gamma_6 \backslash \Omega) \times \Gamma_6\colon\ |k_3^*| \gtrsim N \gg |k_4^*|\};$$
$$A_3 = \{(k,\tau) \in (\Gamma_6 \backslash \Omega) \times \Gamma_6\colon\ |k_4^*| \gtrsim N \gg |k_5^*|\};$$
$$A_4 = \{(k,\tau) \in (\Gamma_6 \backslash \Omega) \times \Gamma_6\colon\ |k_5^*| \gtrsim N\}.$$

It suffices to show that for $j = 1, \dots, 4$,

$$\int_{A_j} \frac{\overline{M}_6(k_1,\dots,k_6)\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_6}(k_6,\tau_6)}{\langle k_1\rangle m(k_1)\langle k_6\rangle m(k_6)} \lesssim \text{RHS of (3.56)}. \tag{3.57}$$

Step 1: estimate in A_1. By Lemma 3.4(2), we have $|\overline{M}_6| \lesssim |k_3^*||k_4^*|$, which implies that

$$\begin{aligned}
\text{LHS of (3.57)} &\lesssim N^{2s-2}\int_{A_1} \frac{\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_6}(k_6,\tau_6)}{|k_1^*|^s|k_2^*|^s\langle k_5^*\rangle\langle k_6^*\rangle} \\
&\sim N^{2s-2}\int_{A_1} |k_1|^{-2s}\cdot \widehat{f_1^*}\widehat{f_3^*}\cdot\widehat{f_2^*}\widehat{f_4^*}\cdot J_x^{-1}\widehat{f_5^*}\cdot J_x^{-1}\widehat{f_6^*} \\
&\lesssim N^{-2}\int_{A_1} \widehat{f_1^*}\widehat{f_3^*}\cdot\widehat{f_2^*}\widehat{f_4^*}\cdot J_x^{-1}\widehat{f_5^*}\cdot J_x^{-1}\widehat{f_6^*},
\end{aligned} \tag{3.58}$$

where f_j^* is one that $\widehat{f_j^*} = \widehat{f_j^*}(k_j^*,\tau_j^*)$. Note that $|k_1^* \pm k_3^*| \sim |k_1^*| \gtrsim N$ and $|k_2^* \pm k_4^*| \sim |k_2^*| \gtrsim N$ in A_1, and by the definitions (2.15) and (2.22), Plancherel's identity, Hölder's inequality, (2.13) and (2.25), we have

$$\begin{aligned}
\text{RHS of (3.58)} &\leqslant N^{-2}\int I_N^{\pm}(f_1^*,f_3^*)\cdot I_N^{\pm}(f_2^*,f_4^*)\cdot J_x^{-1}f_5^*\cdot J_x^{-1}f_6^*\,dx\,dt \\
&\lesssim N^{-2}\|I_N^{\pm}(f_1^*,f_3^*)\|_{L^2_{xt}}\|I_N^{\pm}(f_2^*,f_4^*)\|_{L^2_{xt}}\|J_x^{-1}f_5^*\|_{L^\infty_{xt}}\|J_x^{-1}f_6^*\|_{L^\infty_{xt}} \\
&\lesssim C^2(N,\lambda)N^{-2}\lambda^{0+}\cdot\|f_1\|_{X^+_{0,\frac12+}}\|f_2\|_{X^-_{0,\frac12+}}\cdots\|f_5\|_{X^+_{0,\frac12+}}\|f_6\|_{X^-_{0,\frac12+}} \\
&\lesssim \text{RHS of (3.56)}.
\end{aligned}$$

This implies (3.57) for $j = 1$.

Step 2: estimate in A_2. It vanishes since $A_2 = \emptyset$.

Step 3: estimate in A_3. Note that in this case, we have

$$\text{LHS of (3.57)} \lesssim N^{4s-4}\int_{A_3}\frac{|\overline{M}_6(k_1,\dots,k_6)|\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_6}(k_6,\tau_6)}{|k_1^*|^s|k_2^*|^s|k_3^*|^s|k_4^*|^s\langle k_5^*\rangle\langle k_6^*\rangle}. \tag{3.59}$$

We further split A_3 into two regions and denote

$$A_{31} = \{(k,\tau) \in A_3\colon\ |k_1^*| \gg |k_3^*|\};$$
$$A_{32} = \{(k,\tau) \in A_3\colon\ |k_1^*| \sim |k_3^*|\}.$$

Case 1: *estimate in* A_{31}. By Lemma 3.4(i), we have $|\overline{M}_6| \lesssim m(\xi_1^*)m(\xi_3^*)|k_1^*||k_3^*|$, and note that

$$|k_1^* \pm k_3^*| \sim |k_1^*| \gtrsim N, \qquad |k_2^* \pm k_4^*| \sim |k_2^*| \gtrsim N \quad \text{in } A_{31}.$$

Similar to the estimate in A_1, we have

$$\begin{aligned}
\text{LHS of (3.57)} &\lesssim N^{2s-2} \int\limits_{A_{31}} \frac{\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_6}(k_6,\tau_6)}{|k_2^*|^s|k_4^*|^s\langle k_5^*\rangle\langle k_6^*\rangle} \\
&\lesssim N^{-2}\int I_N^{\pm}(f_1^*, f_3^*)\cdot I_N^{\pm}(f_2^*, f_4^*)\cdot J_x^{-1}f_5^*\cdot J_x^{-1}f_6^* \\
&\lesssim N^{-2}\|I_N^{\pm}(f_1^*, f_3^*)\|_{L^2_{xt}}\|I_N^{\pm}(f_2^*, f_4^*)\|_{L^2_{xt}}\|J_x^{-1}f_5^*\|_{L^\infty_{xt}}\|J_x^{-1}f_6^*\|_{L^\infty_{xt}} \\
&\lesssim \text{RHS of (3.56)}.
\end{aligned}$$

Case 2: *estimate in* A_{32}. Note that $|k_3^*| \sim |k_4^*|$ in $\Gamma_6\backslash\Omega_2$, we have

$$|k_1^*| \sim |k_2^*| \sim |k_3^*| \sim |k_4^*| \quad \text{in } A_{32}.$$

Now we further split A_{32} into three regions,

$$A_{321} = \{(k,\tau) \in A_{32}\colon\ \{k_1^*, k_2^*, k_3^*, k_4^*\} = \{k_1, k_2, k_3, k_4\}\};$$
$$A_{322} = \{(k,\tau) \in A_{32}\colon\ \{k_1^*, k_2^*, k_3^*, k_4^*\} = \{k_1, k_2, k_3, k_5\}\};$$
$$A_{323} = \{(k,\tau) \in A_{32}\colon\ \{k_1^*, k_2^*, k_3^*, k_4^*\} = \{k_1, k_2, k_4, k_6\}\}.$$

Case 2(a): *estimate in* A_{321}. We may assume that $k_1 > 0$ by symmetry. Then one of the following four cases must occur

$$A_{3211} = \{(k,\tau) \in A_{321}\colon\ k_1 > 0,\ k_2 > 0,\ k_3 < 0,\ k_4 < 0\};$$
$$A_{3212} = \{(k,\tau) \in A_{321}\colon\ k_1 > 0,\ k_2 < 0,\ k_3 < 0,\ k_4 > 0\};$$
$$A_{3213} = \{(k,\tau) \in A_{321}\colon\ k_1 > 0,\ k_2 < 0,\ k_3 < 0,\ k_4 < 0\};$$
$$A_{3214} = \{(k,\tau) \in A_{321}\colon\ k_1 > 0,\ k_2 < 0,\ k_3 > 0,\ k_4 < 0\}.$$

Estimate in A_{3211}. By Lemma 3.4(i), we have $|\overline{M}_6| \lesssim m(k_1^*)m(k_3^*)|k_1^*||k_3^*|$, which implies that

$$\begin{aligned}
\text{LHS of (3.57)} &\lesssim N^{2s-2} \int\limits_{A_{3211}} \frac{\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_6}(k_6,\tau_6)}{|k_2^*|^s|k_4^*|^s\langle k_5^*\rangle\langle k_6^*\rangle} \\
&= N^{2s-2}\int\limits_{A_{3211}} |k_2^*|^{-s}|k_4^*|^{-s}\cdot\widehat{f_1}\widehat{f_3}\cdot\widehat{f_2}\widehat{f_4}\cdot\widehat{J_x^{-1}f_5}\cdot\widehat{J_x^{-1}f_6} \\
&\lesssim N^{-2}\int\limits_{A_{3211}} \widehat{f_1}\widehat{f_3}\cdot\widehat{f_2}\widehat{f_4}\cdot\widehat{J_x^{-1}f_5}\cdot\widehat{J_x^{-1}f_6}.
\end{aligned} \tag{3.60}$$

Note that

$$|k_1 - k_3| = |k_1| + |k_3| \gtrsim N, \qquad |k_2 - k_4| = |k_2| + |k_4| \gtrsim N \quad \text{in } A_{3211},$$

then by (2.13) and (2.25), we have,

$$\begin{aligned}
\text{RHS of (3.60)} &\leqslant N^{-2} \int I_N^-(f_1, f_3) \cdot I_N^-(f_2, f_4) \cdot J_x^{-1} f_5 \cdot J_x^{-1} f_6 \, dx\, dt \\
&\lesssim N^{-2} \big\| I_N^-(f_1, f_3) \big\|_{L^2_{xt}} \big\| I_N^-(f_2, f_4) \big\|_{L^2_{xt}} \big\| J_x^{-1} f_5 \big\|_{L^\infty_{xt}} \big\| J_x^{-1} f_6 \big\|_{L^\infty_{xt}} \\
&\lesssim \text{RHS of (3.56)}.
\end{aligned}$$

Estimate in A_{3212}. It is the same as the estimate in A_{3211}.

Estimate in A_{3213}. Note that

$$|k_1 + k_4| \sim |k_2 + k_3| = |k_2| + |k_3| \gtrsim N \quad \text{in } A_{3213}.$$

Therefore, similar to A_{3211}, we have

$$\begin{aligned}
\text{LHS of (3.57)} &\lesssim N^{-2} \big\| I_N^+(f_1, f_4) \big\|_{L^2_{xt}} \big\| I_N^+(f_2, f_3) \big\|_{L^2_{xt}} \big\| J_x^{-1} f_5 \big\|_{L^\infty_{xt}} \big\| J_x^{-1} f_6 \big\|_{L^\infty_{xt}} \\
&\lesssim \text{RHS of (3.56)}.
\end{aligned}$$

Estimate in A_{3214}. We divide it into two parts once again and write

$$\begin{aligned}
A_{3214a} &= \big\{ (k, \tau) \in A_{3214} \colon\ |k_1 + k_2| \lesssim |k_5 + k_6| \big\}; \\
A_{3214b} &= \big\{ (k, \tau) \in A_{3214} \colon\ |k_1 + k_2| \gg |k_5 + k_6| \big\}.
\end{aligned}$$

Estimate in A_{3214a}. By Lemma 3.4(iii) we have

$$|\overline{M}_6| \lesssim m\big(k_1^*\big) \big|k_1^*\big| \big|k_5^*\big|.$$

Therefore, by (3.59), (2.12) and (2.25), we have

$$\begin{aligned}
\text{LHS of (3.57)} &\lesssim N^{3s-3} \int\limits_{A_{3214a}} \frac{\widehat{f_1}(k_1, \tau_1) \cdots \widehat{f_6}(k_6, \tau_6)}{|k_2|^s |k_3|^s |k_4|^s \langle k_6^* \rangle} \\
&\sim N^{3s-3} \int\limits_{A_{3214a}} |k_2|^{-3s+} \cdot \widehat{f_1} \widehat{f_5^*} \cdot \widehat{J_x^{0-} f_2} \cdot \widehat{J_x^{0-} f_3} \cdot \widehat{J_x^{0-} f_4} \cdot \widehat{J_x^{-1} f_6^*} \\
&\lesssim N^{-3+} \big\| I_N^{\pm}\big(f_1, f_5^*\big) \big\|_{L^2_{xt}} \big\| J_x^{0-} f_2 \big\|_{L^6_{xt}} \cdots \big\| J_x^{0-} f_4 \big\|_{L^6_{xt}} \big\| J_x^{-1} f_6^* \big\|_{L^\infty_{xt}} \\
&\lesssim C(N, \lambda) N^{-3+} \lambda^{0+} \cdot \|f_1\|_{X^+_{0,\frac{1}{2}+}} \|f_2\|_{X^-_{0,\frac{1}{2}+}} \cdots \|f_5\|_{X^+_{0,\frac{1}{2}+}} \|f_6\|_{X^-_{0,\frac{1}{2}+}} \\
&\lesssim \text{RHS of (3.56)},
\end{aligned}$$

where at the last step we use the fact that

$$C(N, \lambda) N^{-3+} = N^{-\alpha(s)-3+} \leqslant N^{-2\alpha(s)-2+}.$$

Estimate in A_{3214b}. By Lemma 3.4(iv) we have

$$|\overline{M}_6| \lesssim m(k_1)|k_1||k_1+k_2|.$$

We set $|k_2| \sim N_2$, $|k_1+k_2| \sim N_{12}$ by dyadic decomposition, thus

$$|k_2| \sim |k_3| \sim |k_4| \sim N_2, \qquad |k_1+k_2| \sim |k_3+k_4| \sim N_{12}.$$

Then by (2.12), (2.25) and (3.59), we have

$$\begin{aligned}
\text{LHS of (3.57)} &\lesssim N^{3s-3} \int\limits_{A_{3214b}} \frac{|k_1+k_2|\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_6}(k_6,\tau_6)}{|k_2|^s|k_3|^s|k_4|^s\langle k_5\rangle\langle k_6\rangle} \\
&\lesssim N^{3s-3}N_{12}N_2^{-3s} \int\limits_{A_{3214b}} \widehat{f_1}\widehat{f_2}\cdot\widehat{f_3}\widehat{f_4}\cdot J_x^{-1}\widehat{f_5}\cdot J_x^{-1}\widehat{f_6} \\
&\lesssim N^{3s-3}N_{12}N_2^{-3s} \int I_{N_{12}}^+(f_1,f_2)\cdot I_{N_{12}}^+(f_3,f_4)J_x^{-1}f_5\cdot J_x^{-1}f_6\,dx\,dt \\
&\lesssim N^{-3s-3}N_{12}N_2^{-3s}\big\|I_{N_{12}}^+(f_1,f_2)\big\|_{L^2_{xt}}\big\|I_{N_{12}}^+(f_3,f_4)\big\|_{L^2_{xt}}\big\|J_x^{-1}f_5\big\|_{L^\infty_{xt}}\big\|J_x^{-1}f_6\big\|_{L^\infty_{xt}} \\
&\lesssim N^{-3s-3}N_{12}N_2^{-3s}C^2\big(N_{12}\lambda^{0+},\lambda\big)\|f_1\|_{X^+_{0,\frac12+}}\|f_2\|_{X^-_{0,\frac12+}}\cdots\|f_5\|_{X^+_{0,\frac12+}}\|f_6\|_{X^-_{0,\frac12+}}.
\end{aligned} \tag{3.61}$$

If $N_{12} \gtrsim \lambda$, then $C(N_{12},\lambda) = \lambda^{-\frac12} \leqslant N^{-\alpha(s)}$ and thus

$$\begin{aligned}
N^{-3s-3}N_{12}N_2^{-3s}C^2(N_{12},\lambda)\lambda^{0+} &\leqslant N^{-2\alpha(s)}N^{-3s-3}N_{12}N_2^{-3s}\lambda^{0+} \\
&\lesssim N^{-3s-3-2\alpha(s)}N_2^{1-3s}\lambda^{0+} \\
&\lesssim N^{-2-2\alpha(s)}\lambda^{0+}
\end{aligned}$$

since $s \geqslant \frac13$. On the other hand, if $N_{12} \lesssim \lambda$, then $C(N_{12},\lambda) = N_{12}^{-\frac12}$ and thus

$$\begin{aligned}
N^{-3s-3}N_{12}N_2^{-3s}C^2(N_{12},\lambda)\lambda^{0+} &= N^{-3s-3}N_2^{-3s}\lambda^{0+} \\
&\lesssim N^{-3}\lambda^{0+} \lesssim N^{-2-2\alpha(s)}\lambda^{0+}
\end{aligned}$$

since $\alpha(s) \leqslant \frac12$. Therefore, we obtain the desirable result in (3.57) in A_{3214b} by (3.61). Thus we have finished the proof in A_{321}.

Case 2(b): *estimate in* A_{322}. It vanishes since $A_{322} = \emptyset$.

Case 2(c): *estimate in* A_{323}. In this case, $|k_1| \sim |k_2| \sim |k_4| \sim |k_6|$. Moreover, by Lemma 3.4(i), we have $|\overline{M}_6| \lesssim m^2(k_1)|k_1|^2$, which implies that

$$\begin{aligned}
\text{LHS of (3.57)} &\lesssim N^{2s-2}\int\limits_{A_{323}} \frac{\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_6}(k_6,\tau_6)}{|k_4|^s|k_6|^s\langle k_5^*\rangle\langle k_6^*\rangle}\\
&\lesssim N^{2s-2}\int\limits_{A_{323}} |k_1|^{-2s}\widehat{f_1}\,\widehat{f_6}\cdot\widehat{f_2}\,\widehat{f_4}\cdot J_x^{-1}\widehat{f_3}\cdot J_x^{-1}\widehat{f_5}\\
&\lesssim N^{-2}\int\limits_{A_{323}} \widehat{f_1}\,\widehat{f_6}\cdot\widehat{f_2}\,\widehat{f_4}\cdot\widehat{J_x^{-1}f_3}\cdot\widehat{J_x^{-1}f_5}. \qquad (3.62)
\end{aligned}$$

In $\Gamma_6\backslash\Omega_4$, we have $|k_1|-|k_2|\sim|k_1|$ and the variables k_2,k_4,k_6 are not of the same signs. Without loss of generality, we may assume $k_2\cdot k_4<0$. Then first, we have

$$|f_1+f_6|\geqslant|f_1|-|f_6|\geqslant|f_1|-|f_2|\sim|k_1|\gtrsim N. \qquad (3.63)$$

Second, we have

$$|f_2-f_4|=|f_2|+|f_4|\gtrsim N. \qquad (3.64)$$

Then by (3.63), (3.64) with (2.15) and (2.22), and by (2.12) and (2.25), we have

$$\begin{aligned}
\text{RHS of (3.62)} &\lesssim N^{-2}\int I_N^+(f_1,f_6)\cdot I_N^-(f_2,f_4)\cdot J_x^{-1}f_3\cdot J_x^{-1}f_5\\
&\lesssim N^{-2}\big\|I_N^+(f_1,f_6)\big\|_{L^2_{xt}}\big\|I_N^-(f_2,f_4)\big\|_{L^2_{xt}}\big\|J_x^{-1}f_3\big\|_{L^\infty_{xt}}\big\|J_x^{-1}f_5\big\|_{L^\infty_{xt}}\\
&\lesssim \text{RHS of (3.56)}.
\end{aligned}$$

This yields (3.57) for $j=3$.

Step 4: estimate in A_4**.** By Lemma 3.4(i), we have

$$\begin{aligned}
\text{LHS of (3.57)} &\lesssim N^{3s-3}\int\limits_{A_4}\frac{\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_6}(k_6,\tau_6)}{|k_2^*|^s|k_4^*|^s|k_5^*|^s m(k_6^*)\langle k_6^*\rangle}\\
&\lesssim N^{-3+}\big\|J_x^{0-}f_1^*\big\|_{L^6_{xt}}\cdots\big\|J_x^{0-}f_6^*\big\|_{L^6_{xt}}\\
&\lesssim \text{RHS of (3.56)}.
\end{aligned}$$

This completes the proof of the proposition. □

Proposition 3.3. *For any $s>\frac{1}{5}$, we have*

$$\left|\int_0^\delta \Lambda_{10}(\overline{M}_{10})\,dt\right|\lesssim N^{-2-2\alpha(s)+}\lambda^{0+}\,\|Iu\|^{10}_{X^\delta_{1,\frac12+}}. \qquad (3.65)$$

Proof. Since $\overline{M}_{10}=0$ for $|k_1|,\ldots,|k_{10}|\ll N$, we may assume that $|k_1^*|\sim|k_2^*|\gtrsim N$. Similar to (3.56), it suffices to show that

$$\int_{\Gamma_{10}^2} \frac{\overline{M}_{10}(k_1,\ldots,k_{10})\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_{10}}(k_{10},\tau_{10})}{m(k_1)\langle k_1\rangle\cdots m(k_{10})\langle k_{10}\rangle}$$
$$\lesssim N^{-2-2\alpha(s)+}\lambda^{0+}\|f_1\|_{X^+_{0,\frac12+}}\|f_2\|_{X^-_{0,\frac12+}}\cdots\|f_{2p+1}\|_{X^+_{0,\frac12+}}\|f_{10}\|_{X^-_{0,\frac12+}}, \tag{3.66}$$

where $\Gamma_{10}^2=\{(k_1,\ldots,k_{10},\tau_1,\ldots,\tau_{10})\colon\ k_1+\cdots+k_{10}=0,\ \tau_1+\cdots+\tau_{10}=0\}$. Moreover, by Lemma 3.2, we have $|\overline{M}_{10}|\lesssim 1$, then (3.66) is reduced to show

$$\int_{\Gamma_{10}^2} \frac{\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_{10}}(k_{10},\tau_{10})}{\langle k_1\rangle m(k_1)\cdots\langle k_{10}\rangle m(k_{10})}\lesssim \text{RHS of (3.66)}. \tag{3.67}$$

For simplicity, we may assume that $|k_1|\geqslant\cdots\geqslant|k_{10}|$. By symmetry, we divide Γ_{10}^2 into two regions:

$$B_1=\left\{(k_1,\ldots,k_{10},\tau_1,\ldots,\tau_{10})\in\Gamma_{10}^2\colon\ |k_1|\sim|k_2|\gtrsim N\gg|k_3|\right\};$$
$$B_2=\left\{(k_1,\ldots,k_{10},\tau_1,\ldots,\tau_{10})\in\Gamma_{10}^2\colon\ |k_3|\gtrsim N\right\}.$$

Estimate in B_1. By (2.13) and (2.25), the left-hand side of (3.67), restricted on B_1, is bounded by

$$N^{2s-2}\int_{B_1}\frac{\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_{10}}(k_{10},\tau_{10})}{|k_1|^s|k_2|^s\langle k_3\rangle\cdots\langle k_{10}\rangle}$$
$$\lesssim N^{2s-2}\int_{B_1}|k_1|^{-2s}\widehat{f_1}\widehat{f_3}\cdot\widehat{f_2}\widehat{f_4}\cdot\widehat{J_x^{-1}f_5}\cdots\widehat{J_x^{-1}f_{10}}$$
$$\lesssim N^{-2}\big\|I_N^-(f_1,f_3)\big\|_{L^2_{xt}}\big\|I_N^-(f_2,f_4)\big\|_{L^2_{xt}}\big\|J_x^{-1}f_5\big\|_{L^\infty_{xt}}\cdots\big\|J_x^{-1}f_{10}\big\|_{L^\infty_{xt}}$$
$$\lesssim \text{RHS of (3.66)}.$$

Estimate in B_2. Now the worst case is $|k_j|\gtrsim N$ for any $j=1\ldots10$. We only consider this case and thus for any $s>\frac15$, we have

$$N^{10s-10}\int_{B_2}\frac{\widehat{f_1}(k_1,\tau_1)\cdots\widehat{f_{10}}(k_{10},\tau_{10})}{|k_1|^s\cdots|k_{10}|^s}$$
$$\lesssim N^{-8+}\big\|J_x^{0-}f_1\big\|_{L^6_{xt}}\cdots\big\|J_x^{0-}f_6\big\|_{L^6_{xt}}\big\|J_x^{-\frac12-}f_7\big\|_{L^\infty_{xt}}\cdots\big\|J_x^{-\frac12-}f_{10}\big\|_{L^\infty_{xt}}$$
$$\lesssim \text{RHS of (3.66)},$$

where we have used (2.12) and (2.13) in the last step. This completes the proof of the proposition. □

4. Proof of Theorem 1.1

By Proposition 3.1 and (3.28), the solution of (3.26) exists with the lifetime

$$\delta\sim\lambda^{0-},$$

and we have

$$\|Iu_\lambda\|_{X^\delta_{1,\frac{1}{2}+}} \lesssim 1. \tag{4.68}$$

By (3.39) and (3.40), we have

$$E_I^1\big(u_\lambda(t)\big) = E_I^1(\phi_\lambda) + \Lambda_6(\widetilde{\sigma}_6)(0) - \Lambda_6(\widetilde{\sigma}_6)(t) + \int_0^t \big(\Lambda_6(\overline{M}_6) + \Lambda_{10}(\overline{M}_{10})\big)\,ds.$$

By Lemma 3.2, Proposition 3.2, Proposition 3.3 and (4.68), and a bootstrap argument by choosing N suitable large, we obtain that for any $t \in [0, \delta]$,

$$E_I^1\big(u_\lambda(t)\big) \leqslant 2E_I^1(\phi_\lambda) + 2C_1 N^{0-}\|I\phi_\lambda\|_{H^1}^6 + 2C_3 N^{-2-2\alpha(s)+}.$$

Assume that $2E_I^1(\phi_\lambda) \leqslant C_0$, then for any $t \in [0, \delta]$, $E_I^1(u_\lambda(t)) \leqslant 2C_0$ by choosing N large enough. Repeating this process M times, we obtain

$$E_I^1\big(u_\lambda(t)\big) \leqslant 2E_I^1(u_{0,\lambda}) + 2C_1 N^{0-}\|I\phi_\lambda\|_{H^1}^6 + 2C_3 M N^{-2-2\alpha(s)+}.$$

Therefore, $E_I^1(u_\lambda(t)) \leqslant 2C_0$ provided $M \lesssim N^{2+2\alpha(s)-}$, which implies that the solution u_λ exists on $[0, M\delta] \sim [0, N^{2+2\alpha(s)-}\delta]$. Hence, u exists on $[0, \lambda^2 T]$ with the relation

$$N^{2+2\alpha(s)-} \gtrsim \lambda^2 T \sim N^{\frac{2(1-s)}{s}} T,$$

since $\delta \sim \lambda^{0-}$ is absorbed into N^{0-}. Thus $T = N^{0+}$ as long as

$$2 + 2\alpha(s) > \frac{1-s}{s} \quad \Rightarrow \quad 2 + \frac{1-s}{s} > \frac{1-s}{s}; \qquad 3 > \frac{1-s}{s}.$$

That is $s > \frac{2}{5}$. Therefore, we obtain the global well-posedness in $H^s(\mathbb{R})$ for $s > \frac{2}{5}$ by choosing sufficient large N. This completes the proof of Theorem 1.1.

Acknowledgments

The authors would like to thank the anonymous referee for the valuable advice. Y. Li and Y. Wu were partly supported by the NSF of China (No. 10771074, 10976026). Y. Wu was partly supported by Beijing International Center for Mathematical Research. G. Xu was partly supported by the NSF of China (No. 10801015).

References

[1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I: Schrödinger equation, Geom. Funct. Anal. 3 (1993) 107–156.
[2] J. Bourgain, Remark on normal forms and the "I-method" for periodic NLS, J. Anal. Math. 94 (2004) 127–157.
[3] T. Cazenave, Semilinear Schrödinger Equation, Courant Lect. Notes Math., vol. 10, Amer. Math. Soc., 2003.
[4] T. Cazenave, F.B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation, Nonlinear Anal. 14 (1990) 807–836.
[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Global well-posedness result for Schrödinger equations with derivative, SIAM J. Math. Anal. 33 (2) (2001) 649–669.
[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, A refined global well-posedness result for Schrödinger equations with derivatives, SIAM J. Math. Anal. 34 (2002) 64–86.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, J. Amer. Math. Soc. 16 (2003) 705–749.
[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Multilinear estimates for periodic KdV equations, and applications, J. Funct. Anal. 211 (1) (2004) 173–218.
[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Resonant decompositions and the I-method for cubic nonlinear Schrödinger on $\mathbb{R}^2$, Discrete Contin. Dyn. Syst. 21 (3) (2008) 665–686.
[10] D. De Silva, N. Pavlovic, G. Staffilani, N. Tzirakis, Global well-posedness for the L^2-critical nonlinear Schrödinger equation in higher dimensions, Commun. Pure Appl. Anal. 6 (4) (2007) 1023–1041.
[11] D. De Silva, N. Pavlovic, G. Staffilani, N. Tzirakis, Global well-posedness for a periodic nonlinear Schrödinger equation in 1D and 2D, Discrete Contin. Dyn. Syst. 19 (1) (2007) 37–65.
[12] D. De Silva, N. Pavlovic, G. Staffilani, N. Tzirakis, Global well-posedness and polynomial bounds for the defocusing L^2-critical nonlinear Schrödinger equation in $\mathbb{R}$, Comm. Partial Differential Equations 33 (8) (2008) 1395–1429.
[13] C.E. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc. 9 (2) (1996) 573–603.
[14] R. Killip, T. Tao, M. Visan, The cubic nonlinear Schrödinger equation in two dimension with radial data, J. Eur. Math. Soc. 11 (2009) 1203–1258.
[15] Y. Li, Y. Wu, G. Xu, Low regularity global solutions for the focusing mass-critical NLS in $\mathbb{R}$, SIAM J. Math. Anal. 43 (1) (2011) 322–340.
[16] C. Miao, S. Shao, Y. Wu, G. Xu, The low regularity global solutions for the critical generalized KdV equation, Dyn. Partial Differ. Equ. 7 (3) (2010) 265–288.
[17] T. Tao, M. Visan, X. Zhang, Global well-posedness and scattering for the mass-critical nonlinear Schrödinger equation for radial data in high dimensions, preprint, arXiv:math/0609692.
[18] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. 87 (4) (1983) 567–576.